Economic Interpretation of Lagrange Multipliers in a Lagrangean Decomposition Approach of a Production Planning Problem

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6th BACLOG Workshop – October 5th – 7th, 2009

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Overview

- Lagrangean Decomposition
- LP production planning model
- Two MILP model extensions
- Comparison of two Decomposition approaches
- Summary

Lagrangean Decomposition

- Suppose $\begin{array}{l} \max cx \\ s.t. Ax \le d, Bx \le e \end{array}$
- This is equivalent to $\begin{array}{l} \max cx \\ s.t. Ax \le d, By \le e, x = y \end{array}$
- Lagrangean Decomposition: $\begin{array}{l} \max cx + \lambda(y - x) \\ s.t. Ax \le d, By \le e \end{array}$

$$\Leftrightarrow \quad \begin{array}{l} \max cx - \lambda x \\ s.t. Ax \le d \end{array} + \begin{array}{l} \max \lambda y \\ s.t. By \le e \end{array}$$

- Solve the Lagrangean Dual: $\min_{\lambda} \left\{ \begin{array}{l} \max cx + \lambda(y - x) \\ s.t. Ax \le d, By \le e \end{array} \right\}$

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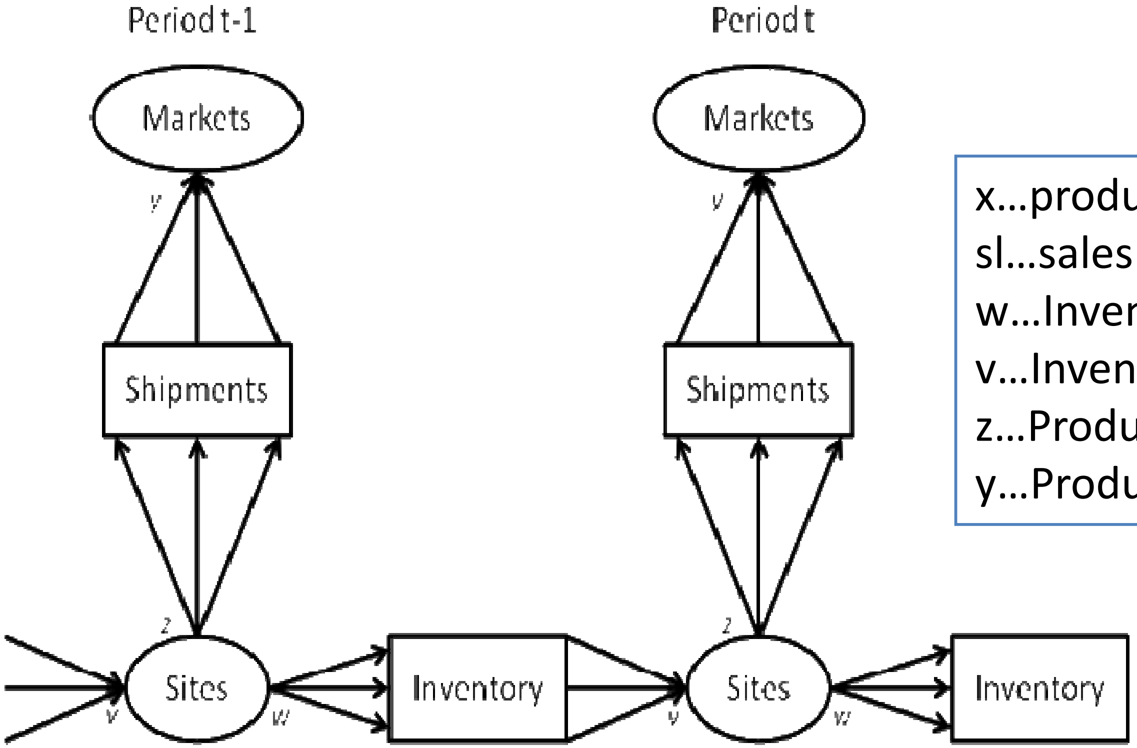
LP Production Planning Model

Illustration of the model with different stages



x…production (cost α)
sl…sales (price β)
w…Inventory for the next period (cost δ)
v…Inventory of the last period (cost δ)
z…Products leaving the sites (cost γ)
y…Products arriving at the markets (cost γ)

LP Production Planning Model

$$\max \quad p = \sum_t \sum_i \left(\sum_m \beta_{mit}\, sl_{mit} - \sum_s \left(\alpha_{sit}\, x_{sit} + \delta_{sit}\, v_{sit} + \sum_m \gamma_{smit}\, y_{smit} \right) \right)$$

$s.t.$

$$(1)\; x_{sit} + v_{sit-1} = w_{sit} + \sum_m z_{smit} \quad \forall s,i,t$$

$$(2)\; \sum_s y_{smit} = sl_{mit} \quad \forall m,i,t$$

$$(3)\; v_{sit} = w_{sit} \quad \forall s,i,t$$

$$(4)\; z_{smit} = y_{smit} \quad \forall s,m,i,t$$

$$(5)\; \sum_i x_{sit} \le pcap_{st} \quad \forall s,t$$

$$(6)\; sl_{mit} \le fcast_{mit} \quad \forall m,i,t$$

$$(7)\; x_{sit}, v_{sit}, w_{sit}, y_{smit}, z_{smit}, sl_{smit} \ge 0 \quad \forall s,m,i,t$$

x…production (cost α)
sl…sales (price β)
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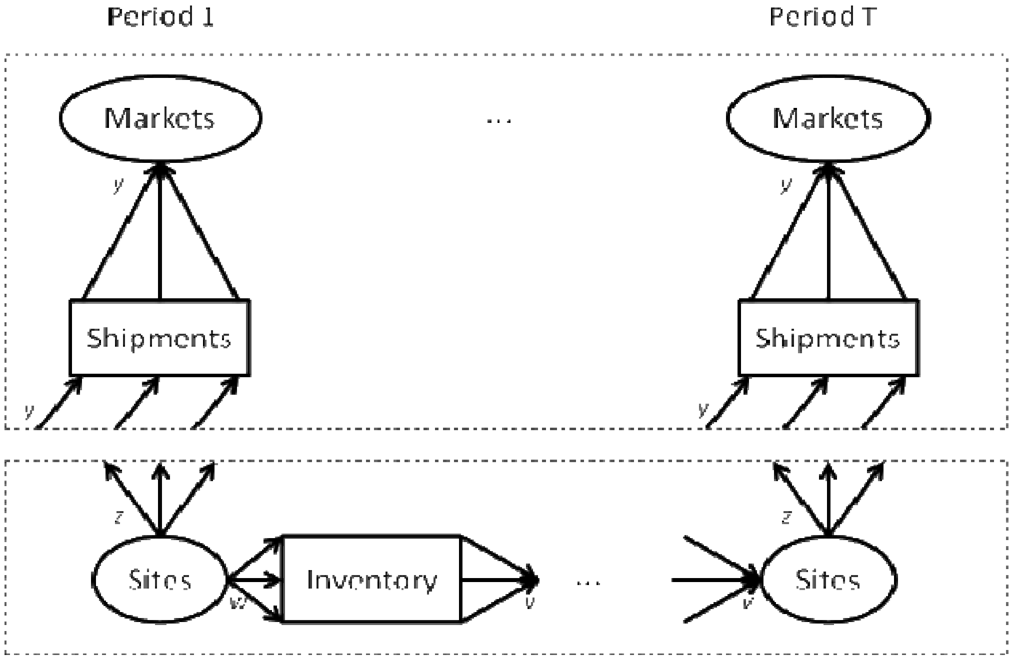
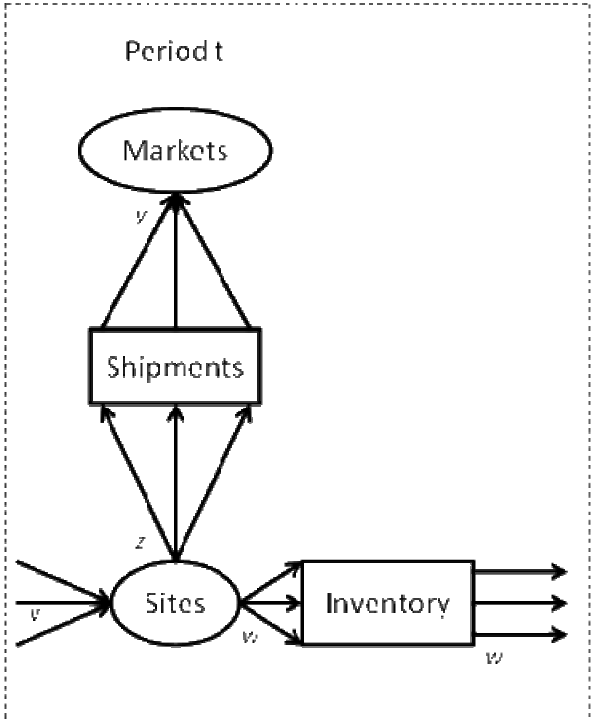
Spatial Decomposition



$$\max p = \sum_t \sum_i \left(\sum_m \beta_{mit} sl_{mit} - \sum_s \left(\alpha_{sit} x_{sit} + \delta_{sit} v_{sit} + \sum_m \gamma_{smit} y_{smit} \right) + \sum_m \sum_s \lambda^{sp}_{smit} \left(z_{smit} - y_{smit} \right) \right)$$

$$s.t.\ (1), (2), (3), (5), (6), (7)\ \forall \mathrm{s, m, i, t}$$

Temporal Decomposition



$$\max p = \sum_t \sum_i \left(\sum_m \beta_{mit} sl_{mit} - \sum_s \left(\alpha_{sit} x_{sit} + \delta_{sit} v_{sit} + \sum_m \gamma_{smit} y_{smit} \right) + \sum_m \sum_s \lambda_{sit}^t \left(w_{smit} - v_{smit} \right) \right)$$

$s.t.\ (1),(2),(4),(5),(6),(7)\ \forall \mathrm{s,m,i,t}$

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Solving the Lagrangean Dual

- Subgradient Method, Cutting Plane Algorithm or hybrid

→*Problem*: Usually computationally expensive to solve

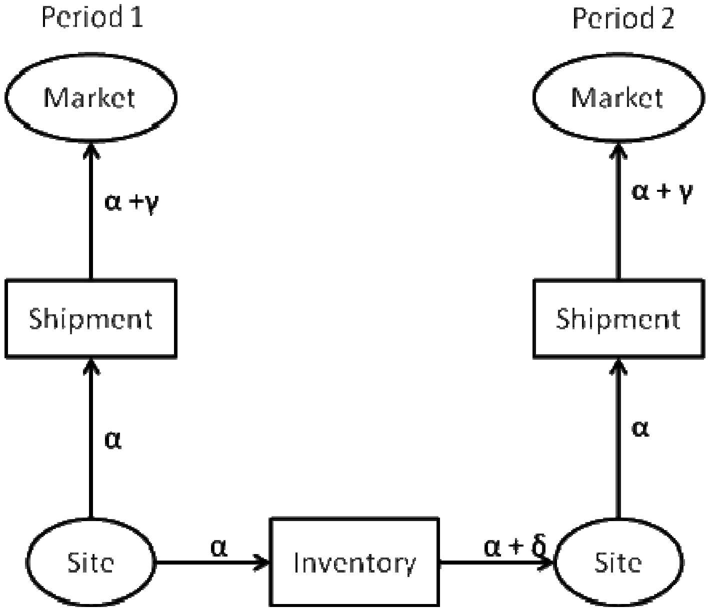
- *Improvement*: Interpret Lagrange multipliers as *Transfer-Prices* between the different stages of the network

→ Exploit this interpretation to **initialize** multipliers

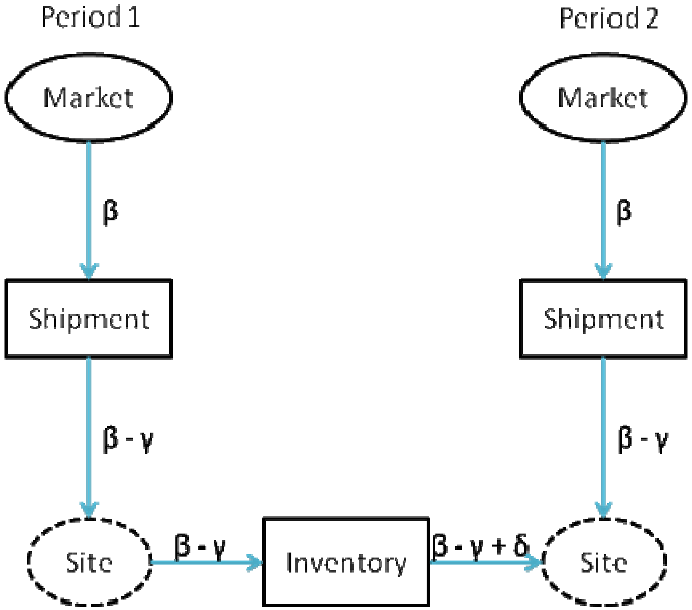
Small-Scale Example

- Bounding scenario determines Lagrange multipliers:

Case 1: Sales bounded by demand fcast.



Case 2: Sales bounded by production cap.



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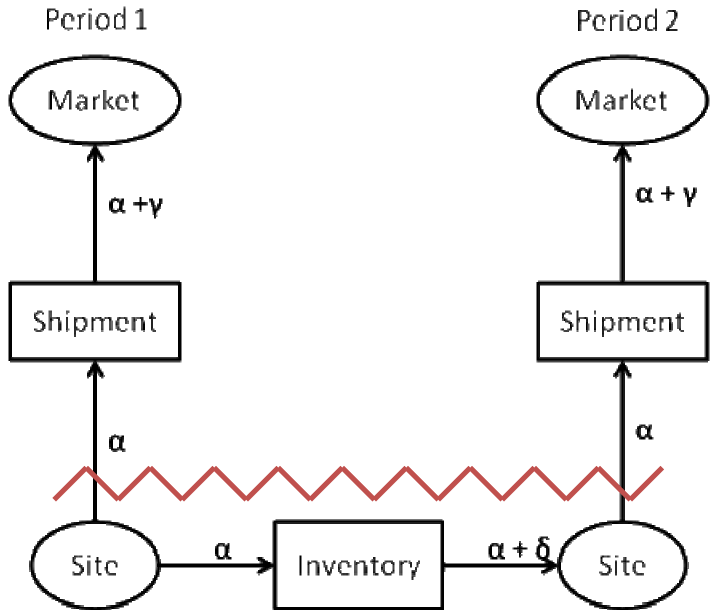
Small-Scale Example (Spatial)

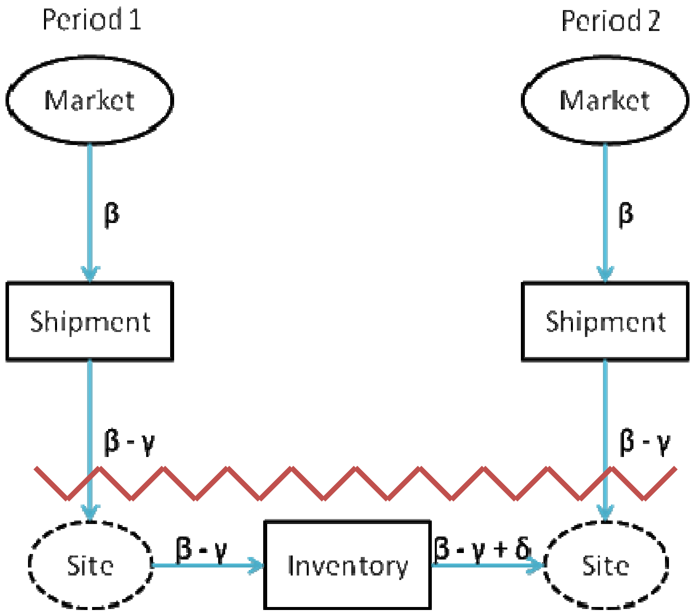
- Bounding scenario determines Lagrange multipliers:

Case 1: Sales bounded by demand fcast.



→Spatial Decomp: $\lambda^{sp}_1 = \alpha = \lambda^{sp}_2$

Case 2: Sales bounded by production cap.

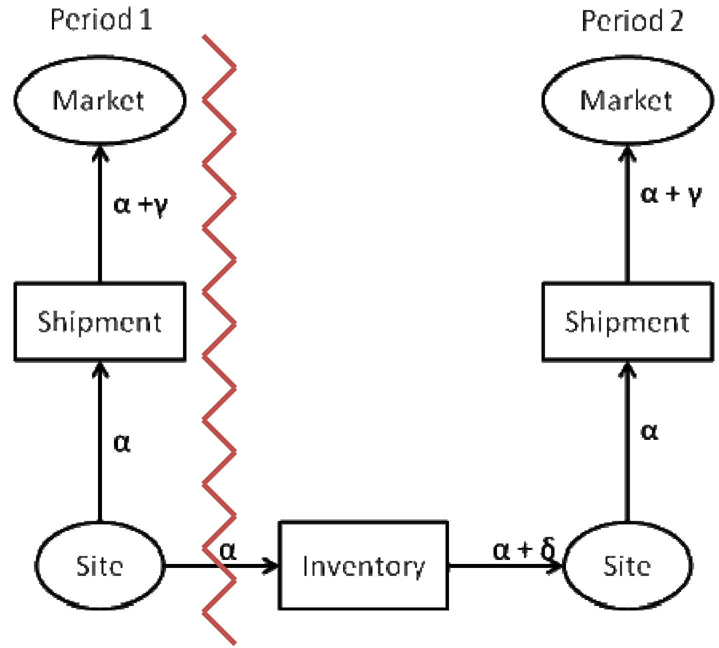


→Spatial Decomp: $\lambda^{sp}_1 = \beta\text{-}\gamma = \lambda^{sp}_2$

Small-Scale Example (Temporal)

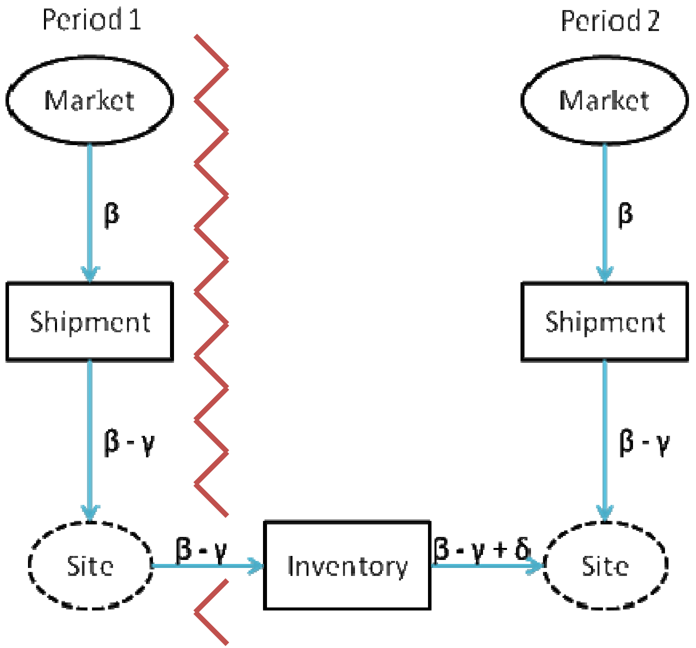
- Bounding scenario determines Lagrange multipliers:

Case 1: Sales bounded by demand fcast.



→*Temporal Decomp:* $\lambda^t = \alpha$

Case 2: Sales bounded by production cap.



→*Temporal Decomp:* $\lambda^t = \beta - \gamma$

→ $\lambda^t = \lambda^{sp}$

Discussion of a Large-Scale Case

1. Multi-period setting: Need to identify bounding scenario

→ *Period-Cut Criterion (Intertemporal product distribution)*

2. High complexity concerning exactly which prices β_{mit} and costs α_{sit}, δ_{sit} and γ_{smit} to chose for specific multipliers

→ *Heuristic approach*

1. Period-Cut

Theorem: If it is profitable to sell products, then if the sales of a period t are bounded by the demand forecast, all following periods t+1,…,T are also forecast bounded.

Corollary: If it is profitable to sell products, then if the sales of a period t are bounded by the production capacity, all previous periods 1,…,t-1 are also capacity bounded.

1. Period-Cut - Proof Outline

Suppose period t_1 is demand forecast bounded and period $t_2 > t_1$ is production capacity bounded.

→ The additional capacity in t_1 would be used to produce products and distribute them to t_2 through inventory

→ Two possible cases:

1. $\sum_{t=1}^{t_2} fcast_t - \sum_{t=1}^{t_2} pcap_t > \sum_{t=1}^{t_1} pcap_t - \sum_{t=1}^{t_1} fcast_t$: t_1 and t_2 are fcast bounded
2. $\sum_{t=1}^{t_2} fcast_t - \sum_{t=1}^{t_2} pcap_t \le \sum_{t=1}^{t_1} pcap_t - \sum_{t=1}^{t_1} fcast_t$: t_1 and t_2 are pcap bounded

2. Tackling the Complexity

- Multi-product, multi-site and multi-market setting

→ *Heuristic approach*: Choose values β_{mit}, α_{sit}, δ_{sit} and γ_{smit} for assigning all Lagrange multipliers that yield a possibly low overall profit, i.e. the worst site-shipment-market combination

Setting the Lagrange Multipliers

- Spatial Decomposition:

$$\lambda_{smit}^{sp} = \min_{m,i}\{\beta_{mit} - \gamma_{smit}\} - \delta_{sit}(t_{2u} - t_{1u}) \forall \text{t until the Period Cut and accordingly set } t_{1u}, t_{2u}$$

$$\lambda_{smit}^{sp} = \max_{s,i}\{\alpha_{sit}\} + \delta_{sit}(t_{2a} - t_{1a}) \forall \text{t after the Period Cut and accordingly set } t_{1a}, t_{2a}$$

- Temporal Decomposition

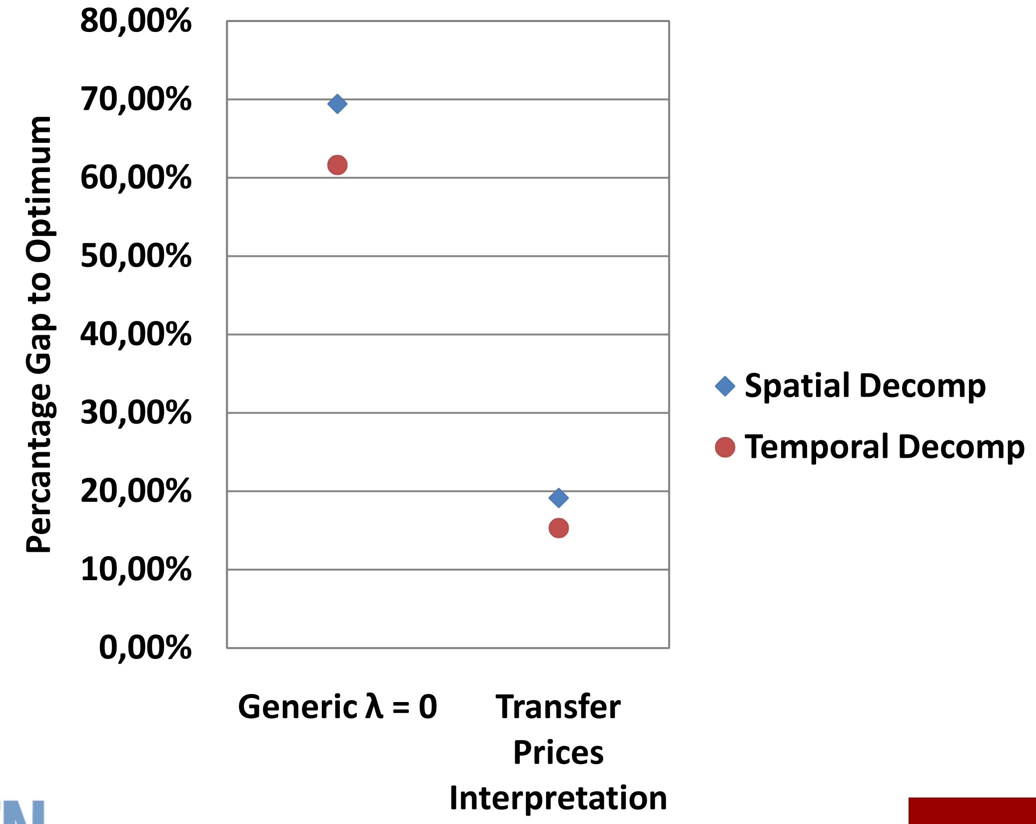
$$\lambda_{sit}^{t} = \min_{m,i}\{\beta_{mit} - \gamma_{smit}\} - \delta_{sit}(t_{2u} - t_{1u}) \forall \text{t until the Period Cut and accordingly set } t_{1u}, t_{2u}$$

$$\lambda_{sit}^{t} = \max_{s,i}\{\alpha_{sit}\} + \delta_{sit}(t_{2a} - t_{1a}) \forall \text{t after the Period Cut and accordingly set } t_{1a}, t_{2a}$$

Note: Inventory cost correction accounts for possible inter-temporal effects

Numerical Results - Improvements

Gap to optimal solution after initializing the multipliers



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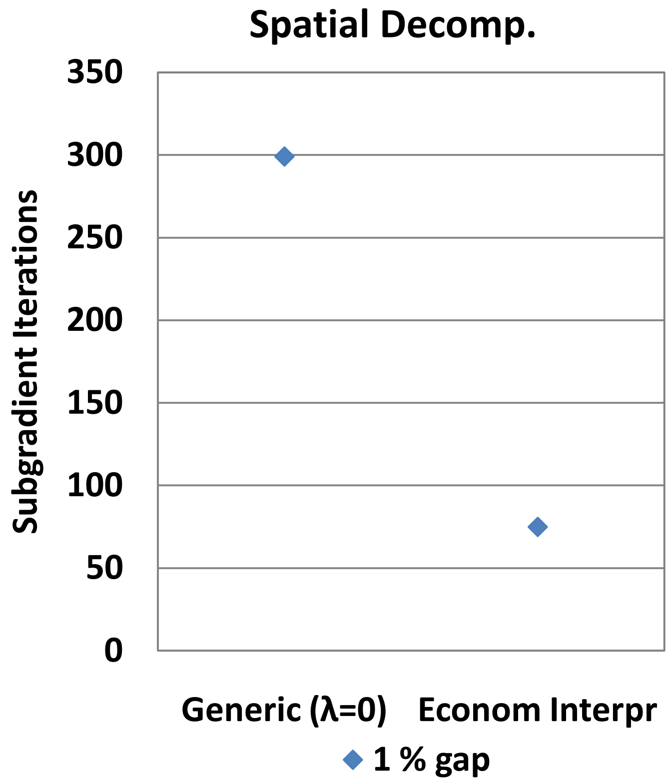
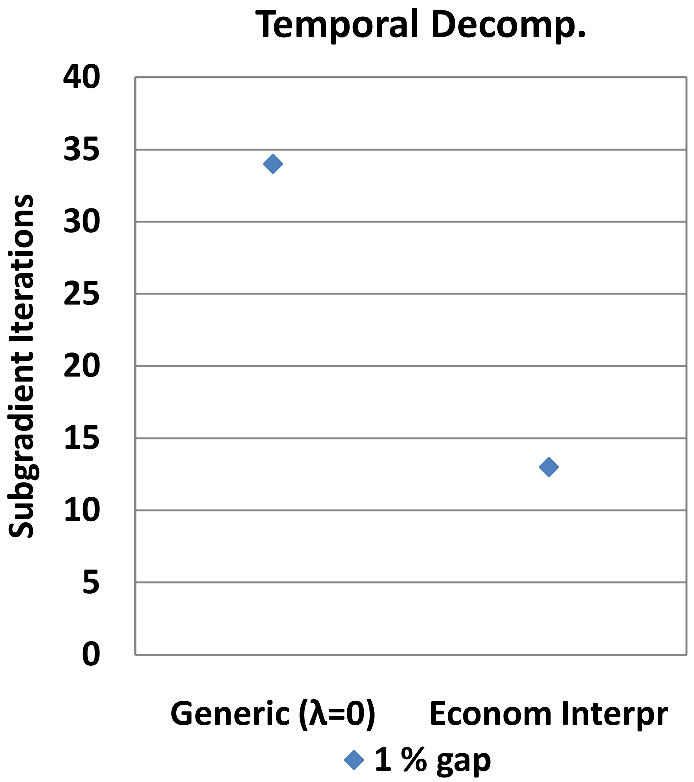
Numerical Results - Improvements

Subgradient iterations until convergence





1st MILP Model Extension

$$\max p = \sum_t \sum_i \left(\sum_m \beta_{mit} sl_{mit} - \sum_s \left(\alpha_{sit} x_{sit} + \delta_{sit} v_{sit} + scst_{sit} setup_{sit} + \sum_m \gamma_{smit} y_{smit} \right) \right)$$

$s.t.$

(1) - (7)

(8) $x_{sit} \leq pcap_{st} setup_{sit} \forall s,i,t$

(9) $\text{setup}_{sit} \in \{0;1\} \forall s,i,t$

x…production (cost α)
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z…Products leaving the sites (cost γ)
y…Products arriving at the markets (cost γ)
setup…Binary setup variable (cost scst)

Note: The Decomposition formulations are not affected by the MILP extension

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Using Economic Insight in MILP Case

- Usage of economic implications to **rigorously bound** the multipliers and cut down the search space

→ Best way: Using marginal values from LP formulation as basis and modify them

LP Basis

Question: What changes the multipliers from the LP model to the MILP formulation?

→ Setup costs, can be seen as increase in production cost

LP Basis - Setting the bounds

$$\lambda_{smit}^{sp} \le \text{LP marginal values of (4)} \; \forall \text{t until the Period Cut}$$

$$\lambda_{smit}^{sp} \ge \max\left\{0, \text{LP marginal values of (4)} - MILPchangeI_{pcap}\right\} \forall \text{t until the Period Cut}$$

$$\lambda_{smit}^{sp} \le \text{LP marginal values of (4)} + MILPchangeI_{fcast} \, \forall \text{t after the Period Cut}$$

$$\lambda_{smit}^{sp} \ge \text{LP marginal values of (4)} \; \forall \text{t after the Period Cut}$$

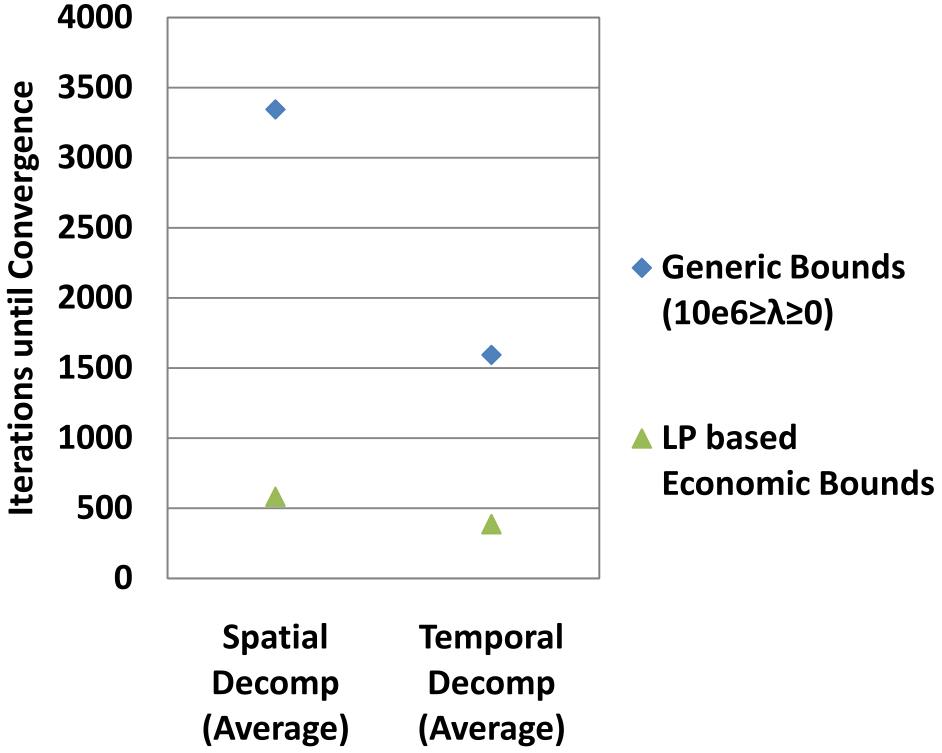
with $$MILPchangeI_{fcast} = \max_{i,t}\left\{\max_{s}\left\{\frac{scst_{sit}}{pcap_{st}}\right\}\right\}$$

$$MILPchangeI_{pcap} = \max_{t}\left\{\max_{s}\left\{\frac{scst_{sit} - scst_{sjt}}{pcap_{st}} \, \forall i, j\right\}\right\}$$

Note for Temporal Decomposition: Marginal values of constraint (3)

Numerical Results 1st MILP

Cutting Plane Algorithm for numerical examples



Gap to full space optimum

Spatial Decomposition	Temporal Decomposition
2.632 %	0.756 %

2^{nd} MILP Model Extension

$$\max p = \sum_t \sum_i \left(\sum_m \beta_{mit} sl_{mit} - \sum_s \left(\alpha_{sit} x_{sit} + \delta_{sit} v_{sit} + scst_{sit} setup_{sit} + \sum_m \gamma_{smit} y_{smit} \right) \right)$$

$s.t.$

$(1),(2),(3),(4),(6),(7),(9)$

$(10)\ x_{sit} = xt_{sit}\, prate_{sit}\ \forall s,i,t$

$(11)\ xt_{sit} \leq setup_{sit} Hortime\, \forall s,i,t$

$(12)\ \sum_i \left(xt_{sit} + setup_{sit} stime_{sit} \right) \leq Hortime\, \forall s,t$

x…production (cost α)
sl…sales (price β)
w…Inventory for the next period (cost δ)
v…Inventory of the last period (cost δ)
z…Products leaving the sites (cost γ)
y…Products arriving at the markets (cost γ)
setup…Binary setup variable (cost scst)
xt…Time spent producing product i in site s

Note: The Decomposition approaches are not affected by the MILP extension

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Relaxed MILP Basis

- Idea very similar to 1st MILP extension: Only monetary change in the MILP are the setup costs

→ *Problem*: Formulation of LP here not possible because of setup times

→ *Solution*: Marginal values for the relaxed constraints from relaxed MILP as basis for the multipliers

rMILP Basis - Setting the bounds

$$\lambda_{smit}^{sp} \geq \max\{0; \text{rMILP marginal values of (4)} - MILPchangeII_{pcap}\}$$

$$\lambda_{smit}^{sp} \leq \text{rMILP marginal values of (4)} + MILPchangeII_{fcast}$$

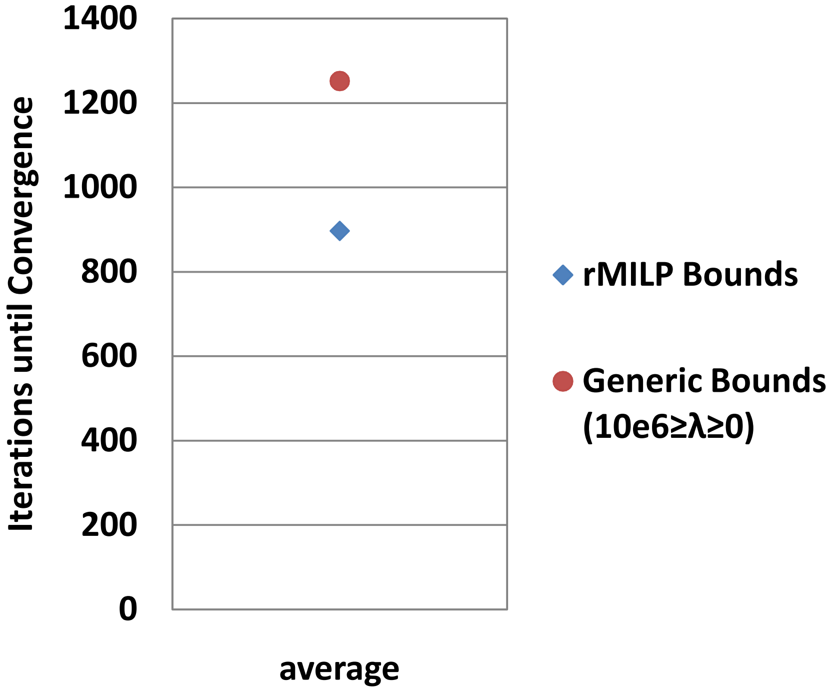
with $$MILPchangeII_{fcast} = \max_{i,t}\left\{\max_{s}\left\{\frac{scst_{sit}}{prate_{sit}\left(Hortime - stime_{sit}\right)}\right\}\right\}$$

$$MILPchangeII_{pcap} = \max_{t}\left\{\max_{s}\left\{\frac{scst_{sit} - scst_{sjt}}{prate_{sit}\left(Hortime - stime_{sit}\right)} \forall i, j\right\}\right\}$$

Note for Temporal Decomposition: Marginal values of constraint (3)

Numerical Results 2nd MILP

Cutting Plane Algorithm for numerical examples



→ 31 % less iterations (~ 50 % less computational time)

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Spatial vs. Temporal Decomposition

Key assumption:

The larger the feasible region of the sales of finished product, the higher the attainable profit

Theorem:

For the mixed integer planning problems presented in this work, temporal decomposition provides a tighter upper bound to the full space optimal profit than spatial decomposition

Spatial vs. Temporal Decomposition

Definitions: Optimal solution of Spatial dual

$$D^s = \{\max p : (sl, x, f, y, setup, v, w) \in S\}$$

$$S = \left\{ \begin{matrix} (sl, x, y, z, setup, v, w) : (sl, x, y, z, setup, v, w) \\ \in Co(FS^{LP}) \cap \{(y, z) : y = z\} \end{matrix} \right\}$$

***FS:** feasible region of the planning problem with **y = z** relaxed*

Spatial vs. Temporal Decomposition

Definitions: Optimal solution of Temporal dual

$$D^t = \{\max p : (sl, x, y, z, setup, v, w) \in T\}$$

$$T = \left\{ \begin{array}{l} (sl, x, y, z, setup, v, w) : (sl, x, y, z, setup, v, w) \\ \in Co(FT^{LP}) \cap \{(v, w) : v = w\} \end{array} \right\}$$

***FT:** feasible region of the planning problem with **v = w** relaxed*

Proof Outline

Proposition 1 (P1)

$$proj_{(sl,setup)} FT^{LP} \subseteq proj_{(sl,setup)} FS^{LP}$$

Corollary

$$proj_{(sl,setup)}\left(Co(FT^{LP})\right) \subseteq proj_{(sl,setup)}\left(Co(FS^{LP})\right)$$

Proposition 2 (P2)

$$proj_{(sl,setup)}\left(Co(FT^{LP})\right) \subseteq proj_{(sl,setup)}\left(Co(FS^{LP}) \cap \{(y,z) : y = z\}\right)$$

Proof Outline

Using P2:

$$proj_{(sl,setup)}\left(Co(FT^{LP})\cap\{(v,w):v=w\}\right)\subseteq proj_{(sl,setup)}\left(Co(FT^{LP})\right)$$

$$\subseteq proj_{(sl,setup)}\left(Co(FS^{LP})\cap\{(y,z):y=z\}\right)$$

Recall:

The larger the feasible region of the sales of finished product, the higher the attainable profit. Thus,

$$\boxed{\begin{aligned} &D^t=\{\max\pi:(\cdot)\in T\}\\ &\leq\\ &D^s=\{\max\pi:(\cdot)\in S\}\end{aligned}}$$

Summary

1. Usage of Decomposition approaches to solve or help solving LP and MILP problems
2. Significant computational savings through initializing or bounding Lagrange multipliers through their dual interpretation
3. Temporal Decomposition approach gives tighter bounds on the full space solution

Thank you for your attention.